The cross sections of the muons and charged pions pairs production at electron-positron annihilation near the threshold

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The processes of muons (tau) and charged pions pairs production at electron-positron annihilation with $\mathcal{O}(\alpha)$ radiative corrections are considered. The calculation results are presented assuming the energies of final particles (cms implied) to be not far significantly from threshold production. The invariant mass distributions for the muon (tau) and pion pairs are obtained both for the initial and final state radiation. Some analytical calculations are illustrated numerically. The pions were assumed to be point-like objects and scalar QED was applied for calculation. The QED radiative corrections related to the final state radiation, additional to the well known Coulomb factor, are treated near threshold region exactly.

I. INTRODUCTION

The current precision of the evaluation of hadron's contribution to anomalous magnetic moment of muon is mainly driven by the systematic error of the cross section of pion pairs production at the region where the total cms energy of pair does not exceed threshold value significantly. Therefore, the lowest order radiative corrections (RC) and effects due to the Coulomb interaction in the final state become essential.

For the purposes of comparisons with experimental data the cases with hard additional

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photon must be calculated in frames of PT. It is a weak point of approaches based on dimensional regularization methods where the separation of soft and hard photon emission can not be arranged. One of the motivations of this paper is to calculate these contributions in frames of traditional QED approach with assigning to photon small mass and calculate the virtual, soft and hard photon contributions separately. In papers [1] the main characteristics of photon emission at annihilation e^+e^- to pair of charged particles was investigated. In papers published in 1983,1985 [2, 3] the spectra and total cross sections were obtained, but the calculation method was too complicated.

Below using the invariant integration method we repeat in part those calculations and obtain the explicit expressions for the spectra distributions on the effective mass of pair and the corresponding contributions to the total cross sections due to photon radiation by initial or final particles. We do not consider the interference of these amplitudes assuming the experimental set-up to be charge-blind. In this case the interference contribution to the total cross section is zero.

In section II, III we consider the final state and initial state radiation of virtual and real photons in muon pair production process. In section IV similar calculations for the charged pion pair production (assuming pion to be point-like object) are done. The results presented in sections II–IV are in agreement with ones obtained in previous papers [2, 3, 4], but have the form more convenient for different applications. Some of them concerning initial state radiation are new ones. In section IV we also discuss some possibilities of experimental separation of contribution of initial and final state radiation. In section V the discussion of accuracy of results obtained is given. Whenever possible, the analytical results are used as a cross-check with ultra relativistic limit.

II. FINAL STATE RADIATION (FSR) IN MUON PAIR PRODUCTION

As well as we are interested in muon effective mass spectrum let us put the cross section in form:

$$d\sigma = \frac{1}{8s} \int \sum_{spins} |M|^2 d\Gamma.$$

The summed over spin states matrix element squared can be put in form (for notations see Fig. 1):

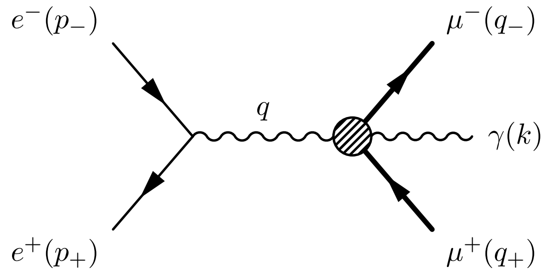

Fig. 1: Final state radiation corrections to $e^+e^- \to \mu^+\mu^-$ process.

$$
\begin{aligned}
&\sum |M|^2 = -(4\pi\alpha)^2 \frac{1}{s^2} L_{\mu\nu} T^{\mu\nu}, \quad s = (p_+ + p_-)^2, \\
&L_{\mu\nu} = Tr\left[\hat{p}_- \gamma_\mu \hat{p}_+ \gamma_\nu\right], \quad T_{\mu\nu} = Tr\left[(\hat{q}_- + M) O_{\mu\eta} (\hat{q}_+ - M) \tilde{O}_{\nu\eta}\right],
\end{aligned} \tag{1}
$$

$$
O_{\mu\nu} = \gamma_\nu \frac{\hat{q}_- + \hat{k} + M}{\chi_-} \gamma_\mu + \gamma_\mu \frac{-\hat{q}_+ - \hat{k} + M}{\chi_+} \gamma_\nu, \tag{2}
$$

where $\chi_\pm = 2kq_\pm$, $\quad p_- + p_+ = q = q_- + q_+ + k$, $\quad q_\pm^2 = M^2$, $\quad p_\pm^2 = m^2$ and $k^2 = 0$. m, M – are electron and muon masses correspondingly. Introducing the energy fractions of final particles we have:

$$
\nu_\pm = \frac{2qq_\pm}{s}; \quad \nu = \frac{2qk}{s}, \quad \nu + \nu_+ + \nu_- = 2,
$$

$$
\int d\Gamma = \int \frac{1}{(2\pi)^5} \frac{d^3 q_-}{2E_-} \frac{d^3 q_+}{2E_+} \frac{d^3 k}{2\omega} \delta^4(p_+ + p_- - q_+ - q_- - k) = \frac{s}{2^7 \pi^3} \int_\Delta^{\beta^2} d\nu \int_{\nu_1}^{\nu_2} d\nu_+,
$$

$$
\nu_{1,2} = \frac{1}{2}(2-\nu) \pm \frac{\nu}{2} R(\nu); \qquad (1-\nu)(1-\nu_-)(1-\nu_+) > \sigma\nu^2,
$$

$$
R(\nu) = \sqrt{1 - \frac{4\sigma}{1-\nu}} = \sqrt{\frac{\beta^2 - \nu}{1-\nu}}, \qquad \beta^2 = 1 - 4\sigma, \qquad \sigma = \frac{M^2}{s}.
$$

Due to gauge invariance of tensor $T^{\mu\nu}$ we can write down the following:

$$
\int d\Gamma\, T_{\mu\nu} = \frac{1}{3}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \int d\Gamma\, T_\eta^\eta.
$$

Further simplification follows from gauge invariance of initial leptons tensor $L^{\mu\nu} q_\mu = 0$. Simple calculation gives

$$
\sum T_\eta^\eta = 4\left[\frac{A}{(1-\nu_+)^2} + \frac{B}{1-\nu_+} + C + (\nu_+ \to \nu_-)\right], \tag{3}
$$

$$A = -\frac{1}{2}(3-\beta^2)(1-\beta^2), \qquad C = -2;$$
$$B = \frac{1}{\nu}(3-\beta^2)(1+\beta^2) - 2(3-\beta^2) + 2\nu.$$

Integration on the muon energy fraction can be performed using the expressions:

$$\int\limits_{\nu_1}^{\nu_2} d\nu_+ \left[\frac{1}{(1-\nu_+)^2};\ \frac{1}{1-\nu_+};\ 1\right] = \left[\frac{1-\nu}{\nu\sigma}R(\nu);\ \ln\frac{1+R(\nu)}{1-R(\nu)};\ \nu R(\nu)\right]. \tag{4}$$

Distribution on the invariant mass square of muons $m_{\mu\mu}^2 = (q_+ + q_-)^2 = s(1-\nu)$ for the case when the energy of hard photon exceeds some value $\omega > \sqrt{s}\Delta/2, \quad \Delta \ll 1$ has a form

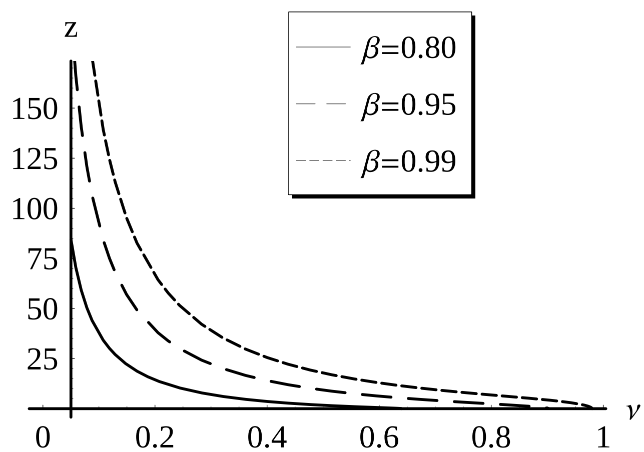

Fig. 2: Distribution on ν for FSR of muons, i.e. the value $z = \left(2\alpha^3/3s\right)^{-1}(d\sigma^h_{FSR}/d\nu)$ (see (5)) is shown.

$$\frac{d\sigma^h_{FSR}}{d\nu} = \frac{2\alpha^3}{3s}\left[\left[\frac{(1+\beta^2)(3-\beta^2)}{\nu} - 2(3-\beta^2) + 2\nu\right]\ln\frac{1+R(\nu)}{1-R(\nu)} - \right.$$
$$\left. -2\left[\frac{3-\beta^2}{\nu}(1-\nu)+\nu\right]R(\nu)\right]. \tag{5}$$

Contribution to the total cross section can be obtained by performing the integration on invariant muon mass. We use the set of integrals:

$$\int\limits_{\Delta}^{\beta^2} R(\nu)\left[\frac{1}{\nu};\ 1;\ \nu\right]d\nu = \left[-L_\beta + \beta\ln\frac{4\beta^2}{(1-\beta^2)\Delta};\ \beta - \frac{1-\beta^2}{2}L_\beta;\right.$$
$$\left.\beta\frac{3-\beta^2}{4} - \frac{(3+\beta^2)(1-\beta^2)}{8}L_\beta\right] + O(\Delta), \tag{6}$$

$$\int_{\Delta}^{\beta^2} \ln\frac{1+R(\nu)}{1-R(\nu)}\left[\frac{1}{\nu};\ 1;\ \nu\right]d\nu = \left[L_\beta \ln\frac{1}{\Delta}+2\Phi(\beta);\ \ -\beta+\frac{1}{2}(1+\beta^2)L_\beta;\right.$$
$$\left.\frac{1}{16}(3+2\beta^2+3\beta^4)L_\beta-\frac{3}{8}\beta(1+\beta^2)\right]+O(\Delta), \tag{7}$$

with

$$L_\beta=\ln\frac{1+\beta}{1-\beta};\quad \Phi(\beta)=\mathrm{Li}_2\left(1-\beta\right)-\mathrm{Li}_2\left(1+\beta\right)-\mathrm{Li}_2\left(\frac{1-\beta}{2}\right)+\mathrm{Li}_2\left(\frac{1+\beta}{2}\right). \tag{8}$$

The result is

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma} \;=\; \frac{2\alpha}{\pi}\sigma_B(s)\left[(\frac{1+\beta^2}{2\beta}L_\beta-1)\ln\frac{1}{\Delta}+\frac{7}{4}-\ln\frac{4\beta^2}{1-\beta^2}-\right.$$
$$\left.-\frac{3(1+\beta^2)}{8(3-\beta^2)}+\frac{9-2\beta^2+\beta^4}{16\beta(3-\beta^2)}L_\beta+\frac{1+\beta^2}{\beta}\Phi(\beta)\right], \tag{9}$$

where $\sigma_B(s)=2\pi\alpha^2\beta(3-\beta^2)/(3s)$ is the cross section of muon pair production in Born approximation. In the ultra relativistic limit we have

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma}\Big|_{\beta\to1}=\frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}[(l_\mu-1)\ln\frac{1}{\Delta}-\frac{3}{4}l_\mu+\frac{11}{8}-\xi_2], \tag{10}$$

where $l_\mu=\ln(s/M^2)$, $\xi_2=\pi^2/6$. This result differs from one given in [5]. The contribution of soft real photons emission with energy $\omega=\sqrt{k^2+\lambda^2}<\sqrt{s}\Delta/2$, where λ - is "photon mass", is given by:

$$\sigma_{FSR}^s=\sigma_B(s)\left(-\frac{\alpha}{4\pi^2}\right)\int\frac{d^3k}{\omega}\left(\frac{q_-}{q_-k}-\frac{q_+}{q_+k}\right)^2,$$

and performing the standard calculations can be written in form [6]:

$$\begin{aligned}\sigma_{FSR}^s \;=\; &\frac{2\alpha}{\pi}\sigma_B(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta-1\right)\left(\ln\frac{M}{\lambda}+\ln\Delta\right)+\right.\\ &\frac{1+\beta^2}{2\beta}\left[\frac{1}{4}L_\beta^2-\mathrm{Li}_2\left(\beta\right)+\mathrm{Li}_2\left(-\beta\right)-\mathrm{Li}_2\left(\frac{1-\beta}{2}\right)-\right.\\ &\left.\ln\left(\frac{1+\beta}{2}\right)\ln(1-\beta)+\frac{1}{2}\ln^2(1+\beta)+\mathrm{Li}_2\left(\frac{1}{2}\right)+L_\beta\ln\frac{2}{1+\beta}\right]+\\ &\left.\ln\left(\frac{1+\beta}{2}\right)+\frac{1-\beta}{2\beta}L_\beta\right].\end{aligned} \tag{11}$$

The correction of virtual photon emission include the Dirac and Pauli formfactors of muon. It has a form [7]:

$$\begin{aligned}
\sigma^{v}_{FSR} &= \frac{2\alpha}{\pi}\sigma_B(s)[(1-\frac{1+\beta^2}{2\beta}L_\beta)\ln\frac{M}{\lambda} - \\
& 1+(\frac{1+\beta^2}{2\beta}-\frac{1}{4\beta})L_\beta+\frac{1+\beta^2}{2\beta}[2\xi_2-\frac{1}{4}L^2_\beta - \\
& L_\beta\ln\frac{2\beta}{1+\beta}+\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)] - \frac{3(1-\beta^2)}{4\beta(3-\beta^2)}L_\beta]. \qquad (12)
\end{aligned}$$

The sum of contributions from virtual and soft real photons reads to be:

$$\begin{aligned}
\sigma^{v+s}_{FSR} &= \frac{2\alpha}{\pi}\sigma_B(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta-1\right)\ln\Delta-1+\ln\frac{1+\beta}{2}+\right.\\
&+\left(\frac{3-2\beta+2\beta^2}{4\beta}-\frac{3(1-\beta^2)}{4\beta(3-\beta^2)}\right)L_\beta+\\
&\left.+\frac{1+\beta^2}{2\beta}\left(-2\mathrm{Li}_2\left(\beta\right)+2\mathrm{Li}_2\left(-\beta\right)+\mathrm{Li}_2\left(\frac{1+\beta}{2}\right)-\mathrm{Li}_2\left(\frac{1-\beta}{2}\right)+3\xi_2\right)\right]. \qquad (13)
\end{aligned}$$

In ultra relativistic limit we have:

$$\sigma^{v+s}_{FSR}\big|_{\beta\to1} = \frac{2\alpha}{\pi}\sigma_B(s)\left[(l_\mu-1)\ln\Delta-1+\frac{3}{4}l_\mu+\xi_2\right]. \qquad (14)$$

The total sum of contributions from virtual, soft and hard real photons does not contain photon mass λ and the separation parameter Δ:

$$\sigma^{e^+e^-\to\mu^+\mu^-\gamma}_{FSR} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta^{\mu^+\mu^-}_{FSR}(\beta), \qquad (15)$$

where

$$\begin{aligned}
\Delta^{\mu^+\mu^-}_{FSR}(\beta) &= \frac{3(5-3\beta^2)}{8(3-\beta^2)}+\frac{(1-\beta)(33-39\beta-17\beta^2+7\beta^3)}{16\beta(3-\beta^2)}L_\beta+\\
&+\ 3\ln\left(\frac{1+\beta}{2}\right)-2\ln\beta+\frac{1+\beta^2}{2\beta}F(\beta), \qquad (16)\\
F(\beta) &= -2\mathrm{Li}_2\left(\beta\right)+2\mathrm{Li}_2\left(-\beta\right)-2\mathrm{Li}_2\left(1+\beta\right)+2\mathrm{Li}_2\left(1-\beta\right)\\
&+\ 3\mathrm{Li}_2\left(\frac{1+\beta}{2}\right)-3\mathrm{Li}_2\left(\frac{1-\beta}{2}\right)+3\ \xi_2. \qquad (17)
\end{aligned}$$

The quantity $\Delta^{\mu^+\mu^-}_{FSR}(\beta)$ agrees with the result obtained in [8] and presented in Fig. 3 as a function of β. This correction in ultra relativistic limit tends to the value 3/8.

$$\sigma^{e^+e^-\to\mu^+\mu^-\gamma}_{FSR}|_{\beta\to1} = \frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}\frac{3}{8} = \frac{\alpha^3}{s}.$$

Cancellation of "large" logarithms $l_\mu = \ln(s/M^2)$ is the consequence of Kinoshita-Lee-Nauenberg theorem [9].

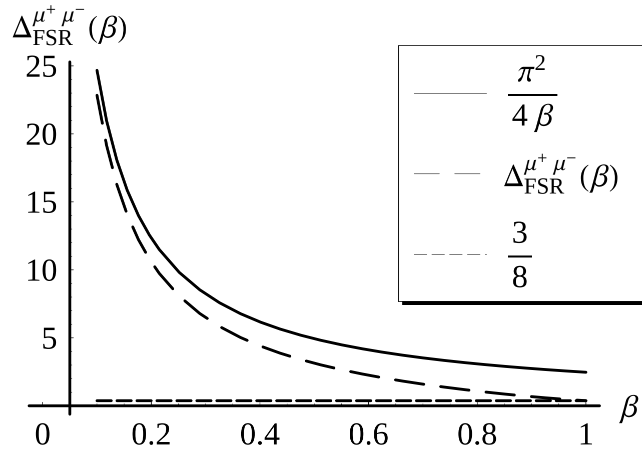

Fig. 3: The dependence of quantity $\Delta_{FSR}^{\mu^+\mu^-}(\beta)$ as a function of β for FSR of muons. See formula (16) and it's asymptotic behavior.

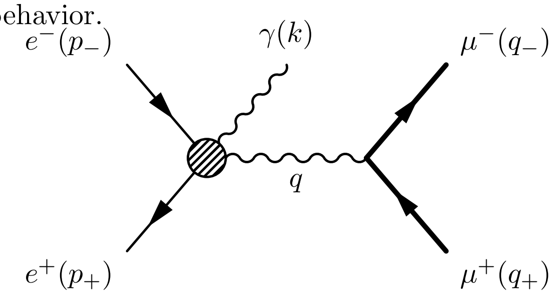

Fig. 4: Initial state radiation corrections to $e^+e^- \to \mu^+\mu^-$ process.

III. INITIAL STATE RADIATION (ISR) IN MUON PAIR PRODUCTION

Matrix element of the process of muon pair production with hard photon radiated from initial state has a form (for notations see Fig. 4):

$$M_{ISR} = \frac{(4\pi\alpha)^{3/2}}{s(1-\nu)} \bar{v}(p_+) \left[\hat{Q} \frac{\hat{p}_- - \hat{k} + m}{-2kp_-} \hat{e}(k) + \hat{e}(k) \frac{-\hat{p}_+ + \hat{k} + m}{-2kp_+} \hat{Q} \right] u(p_-), \tag{18}$$

with $Q_\eta = \bar{u}(q_-)\gamma_\eta v(q_+)$ is the muon current. Using the gauge condition for muon current $q^\eta Q_\eta = 0,\ \ q = q_+ + q_- = p_+ + p_- - k$ we have:

$$\sum \int Q_\mu (Q_\nu)^* \frac{d^3 q_+}{2E_+} \frac{d^3 q_-}{2E_-} \delta^4(q - q_+ - q_-) = D \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \tag{19}$$

$$D = -\frac{2\pi s}{3} \left[\frac{3-\beta^2}{2} - \nu \right] R(\nu), \qquad q^2 = s(1-\nu),$$

with notations given above. Using these relations, the calculation of the summed upon spin states of matrix element squared is straightforward. Performing the angular integrations

$$\int\limits_{-1}^{1} dc [\frac{1}{1-\beta_e c}; \frac{4m^2}{s(1-\beta_e c)^2}; 1] = [l_e; 2; 2], \qquad l_e = \ln\frac{s}{m^2}, \qquad \beta_e = \sqrt{1 - \frac{4m^2}{s}}, \tag{20}$$

we obtain the distribution on the muons invariant mass (see Fig. 5):

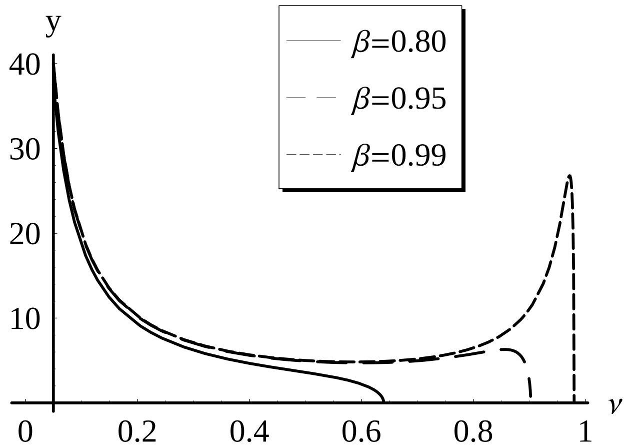

Fig. 5: Distribution of muon pairs as a function on ν for ISR. Vertical axis represents the quantity $y = [(4\alpha^3/3s)(l_e-1)]^{-1}(d\sigma^h_{ISR}/d\nu)$ (see (21)), horizontal axis - the fraction of radiated photon energy ν.

$$\frac{d\sigma^h_{ISR}}{d\nu} = \frac{4\alpha^3}{3s\nu(1-\nu)^2}[1+(1-\nu)^2](l_e-1)\left(\frac{3-\beta^2}{2}-\nu\right)R(\nu), \quad \nu > \Delta. \tag{21}$$

Further integration on the photon energy fraction ν can be performed using the set of integrals given above and two additional ones:

$$\int\limits_0^{\beta^2} R(\nu)[\frac{1}{(1-\nu)^2};\frac{1}{1-\nu}]d\nu = [\frac{2\beta^3}{3(1-\beta^2)};-2\beta+L_\beta].$$

As a result we obtain the cross section due to radiation of hard photon from ISR:

$$\sigma^h_{ISR} = \frac{2\alpha}{\pi}\sigma_B(s)(l_e-1)\left[\ln\frac{1}{\Delta}-\frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta-\frac{4}{3}+2\ln\frac{2\beta}{1+\beta}\right]. \tag{22}$$

The contribution to the cross section taking into account the virtual and soft photons to the initial state is given by:

$$\sigma^{s+v}_{ISR} = \frac{2\alpha}{\pi}\sigma_B(s)[(l_e-1)\ln\Delta+\frac{3}{4}l_e-1+\xi_2]. \tag{23}$$

Let us note that the spectral distribution on invariant mass of final system have a form consistent with renormalization group prescriptions, namely one can recognize the kernel of evolution equation contribution (see (21), (23)).

Now, we can collect all the terms mentioned above and to write out the expression for the total cross section due to ISR:

$$\sigma_{ISR}^{s+v+h} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta_{ISR}^{\mu^+\mu^-}(\beta), \tag{24}$$

$$\Delta_{ISR}^{\mu^+\mu^-}(\beta) = (l_e-1)\left[-\frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta-\frac{4}{3}+2\ln\frac{2\beta}{1+\beta}\right]+\frac{3}{4}l_e-1+\xi_2. \tag{25}$$

The dependence of this quantity on muon's velocity β is shown in Fig. 6. In ultra relativistic

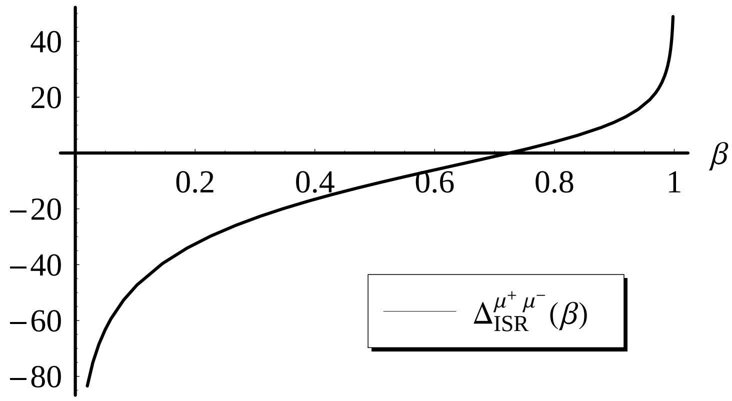

Fig. 6: Distribution of muon pairs on β for ISR. See formula (25) for the quantity $\Delta_{ISR}^{\mu^+\mu^-}(\beta)$.

limit we have:

$$\sigma_{ISR+FSR}^{s+v+h}|_{\beta\to1} = \frac{8\alpha^3}{3s}[\frac{1}{2}l_e l_\mu - \frac{1}{2}l_\mu - \frac{7}{12}l_e + \xi_2 + \frac{17}{24}], \tag{26}$$

which is in agreement with [2, 3]. Leading term $\sim l_e l_\mu$ is in agreement with the result [5].

The total cross section contains the so called double-logarithmical terms ($\sim l_e l_\mu$), which already contradict the renormalization group predictions (single-logarithmic).

IV. THE FINAL STATE RADIATION IN PION PAIR PRODUCTION

It is worth to remind that the total cross section $\sigma(e^+e^-\to\pi^+\pi^-)$ with $\mathcal{O}(\alpha)$ corrections is required in many subjects of particle physics. Particularly it is required to determine with a better accuracy the precision of the evaluation of vacuum polarization effects in photon propagator. The other well known application is the calculation of the hadronic contribution to the anomalous magnetic moment of muon a_μ^{hadr} [10]:

$$a_\mu^{hadr} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_{4M_\pi^2}^{\infty} ds\ \frac{R(s)K(s)}{s}, \qquad R(s) = \frac{\sigma^{e^+e^-\to\pi^+\pi^-}(s)}{\sigma^{e^+e^-\to\mu^+\mu^-}(s)}. \tag{27}$$

A contribution to this integral coming from high energy region can be calculated within QCD framework, while for the low energy range the experimental values R(s) have to be taken as an input [11]. A numerical evaluation of this integral in relative unities gives the value ~ 70 ppm.

The goal of the new experiment at BNL (E969) is to measure the anomalous magnetic moment of muon with the relative accuracy of about ~ 0.25 ppm and to improve the previous result [12] by a factor of two. It follows that the value a_μ^{hadr} should be calculated as precisely as possible. In this context the required theoretical precision of the cross sections with radiative corrections (RC) as well as the calculation accuracy of the vacuum polarization effects should be not worse than $\sim 0.2\%$ as it follows from the estimation: 70 ppm$\times$0.2% $\sim$0.14 ppm. This short observation shows why high precision calculation of the hadronic cross sections are extremely important.

A. Final state radiation

As well as it was done for the muons, the contributions with one photon radiation in the final state can be divided into three separate parts: virtual, soft and hard. The expression for the virtual photon emission from final state can be found in [13] and is given by

$$\begin{aligned}\sigma_v &= \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Bigg[2\ln\frac{M_\pi}{\lambda}\left(1-\frac{1+\beta^2}{2\beta}L_\beta\right)-2+\frac{1+\beta^2}{\beta}L_\beta \\ &+ \frac{1+\beta^2}{\beta}\left(-\frac{1}{4}L_\beta^2+L_\beta\ln\frac{1+\beta}{2\beta}+2\xi_2+\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right)\Bigg]. \end{aligned} \tag{28}$$

Here L_β, λ, β were defined above, β is a pion velocity in c.m. frame, $\sigma_B^{\pi^+\pi^-}(s) = (\pi\alpha^2\beta^3)/(3s)|F_\pi(s)|^2$ is the cross section production of charged pion pair in the Born approximation, $F_\pi(s)$ - pion strong interaction formfactor. The cross section is due to emission of soft photon when its energy does not exceed $\Delta\varepsilon$ is given by:

$$\begin{aligned}\sigma_{FSR}^s &= \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Bigg[2\ln\left(\frac{2\Delta\varepsilon}{\lambda}\right)\left(\frac{1+\beta^2}{2\beta}L_\beta-1\right)+\frac{1}{\beta}L_\beta \\ &+ \frac{1+\beta^2}{\beta}\left(-\frac{1}{4}L_\beta^2+L_\beta\ln\frac{1+\beta}{2\beta}-\xi_2+\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right)\Bigg], \quad \Delta\varepsilon \ll \varepsilon = \frac{\sqrt{s}}{2}. \end{aligned} \tag{29}$$

The sum of the contributions from virtual and soft photons can be presented in convenient way as:

$$\sigma_{FSR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta-1\right)\ln\Delta+b(s)\right], \tag{30}$$

where

$$b(s) = -1 + \frac{1-\beta}{2\beta}\rho + \frac{2+\beta^2}{\beta}\ln\frac{1+\beta}{2} + \frac{1+\beta^2}{2\beta}\left[\rho + \xi_2 + L_\beta \ln\frac{1+\beta}{2\beta^2} + 2\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right],$$

$$\rho = \ln\frac{4}{1-\beta^2}, \qquad \Delta = \frac{\Delta\varepsilon}{\varepsilon}.$$

Calculations similar to ones given above for muons FSR lead to the pion pair invariant mass distribution ($m^2_{\pi\pi} = s(1-\nu)$, see Fig.7):

$$\frac{\sigma^h_{FSR}}{d\nu} = \frac{2\alpha^3\beta^2}{3s}\left[\left(\frac{\nu}{\beta^2} - \frac{1-\nu}{\nu}\right)R(\nu) + \left(\frac{1+\beta^2}{2\nu} - 1\right)\ln\frac{1+R(\nu)}{1-R(\nu)}\right]|F_\pi(s)|^2. \quad (31)$$

Contribution to the total cross section can be obtained performing the integration on

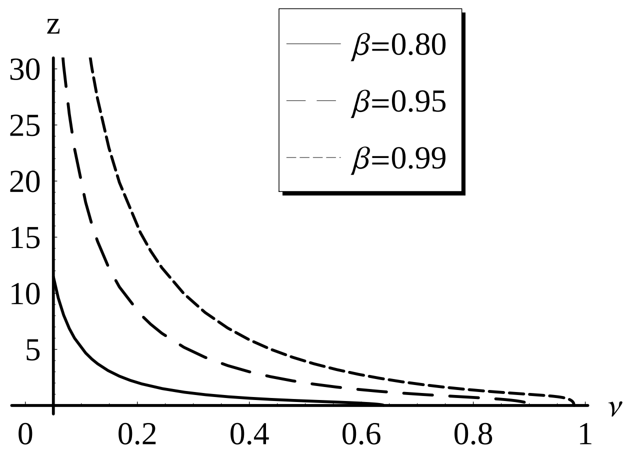

Fig. 7: The pion invariant mass distribution on ν for FSR. The vertical and horizontal axes represent the value $z = [(2\alpha^3/3s)]^{-1}(d\sigma^h_{FSR}/d\nu)$ (see (31)) and fraction of photon energy, respectively.

invariant pion pair mass. In agreement with [4, 14] the relevant contribution has a form:

$$\sigma^h_{FSR} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left[\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right) + 2 + \frac{3-\beta^2}{4\beta^2} - \frac{(3+\beta^2)(1-\beta^2)}{8\beta^3}L_\beta - \ln\frac{4\beta^2}{1-\beta^2} + \frac{1+\beta^2}{\beta}\Phi(\beta)\right], \quad (32)$$

with $\Phi(\beta)$ defined in (8). Now we can write down the complete expression for the total cross section:

$$\sigma^{e^+e^-\to\pi^+\pi^-\gamma}_{FSR} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Delta^{\pi^+\pi^-}_{FSR}(\beta), \quad (33)$$

$$\Delta^{\pi^+\pi^-}_{FSR}(\beta) = \frac{3(1+\beta^2)}{4\beta^2} - 2\ln\beta + 3\ln\frac{1+\beta}{2} + \frac{1+\beta^2}{2\beta}F(\beta) + \frac{(1-\beta)(-3-3\beta+7\beta^2-5\beta^3)}{8\beta^3}L_\beta, \quad (34)$$

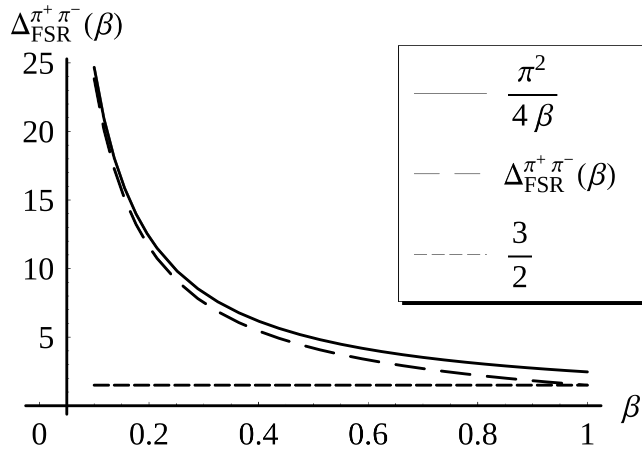

Fig. 8: The dependence of quantity $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ on β for FSR of pions. See formula (34) and it's asymptotic behavior.

with the same expression for $F(\beta)$ as in muon case (17). The factor $\Delta_{FSR}^{\pi^+\pi^-}$ represents the correction to the Born cross section caused by final state radiation. In low β limit $\Delta_{FSR}^{\pi^+\pi^-}(\beta) \approx \pi^2/4\beta$, which is the manifestation of Coulomb interaction of pions. There is exactly the same behavior for $\Delta_{FSR}^{\mu^+\mu^-}(\beta)$ (see (16)). It's well known that in limit $\alpha/\beta \geq 1$ the perturbative analysis is not valid. The relevant modifications of formulae will be given in conclusion.

In ultra relativistic limit we have $\Delta_{FSR}^{\pi^+\pi^-}(\beta \to 1) = 3/2$. One can see again, that all "large" logarithms cancelled out in accordance with Kinoshita-Lee-Nauenberg theorem. Expression for $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ coincide with one obtained in [4, 15]. It is worth noticing that in papers [4, 14, 15] the quantity $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ was presented without separator Δ between soft and hard photons. But for some applications it can be useful to have these two parts separately.

In order to check experimentally the validity of point-like pions assumption, which is used in the paper, it's necessary to separate out the FSR events. Unfortunately we should notice that ISR events 10 times more probable then the FSR ones. Nevertheless there are at least two ways to select FSR events and to suppress the ISR background.

Firstly we may consider the region of ρ-meson peak left slope, i.e. $\sqrt{s} < 770$ MeV. In that case the resonance returning mechanism does not take place and the ratio of FSR events increases.

Second way is to throw out the events with pions acollinearity bigger than some predefined angle, for instance 0.25 rad [16].

Figure 9 shows the result of modelling of value $\sigma_{ISR+FSR}/\sigma_{ISR}$ with application of both FSR separation methods described above. The different curves correspond to different energy thresholds of emitted photons ($\omega > 10-170$ MeV). One can see that the energy range from 720 to 780 MeV is preferable for our purpose - if photon energy exceeds 150 MeV then the ratio $\sigma_{ISR+FSR}/\sigma_{ISR}$ is about 5, this means that the relative admixture of ISR events is about 20 % only. It is worth to notice that the form of spectrum at high photon energies is

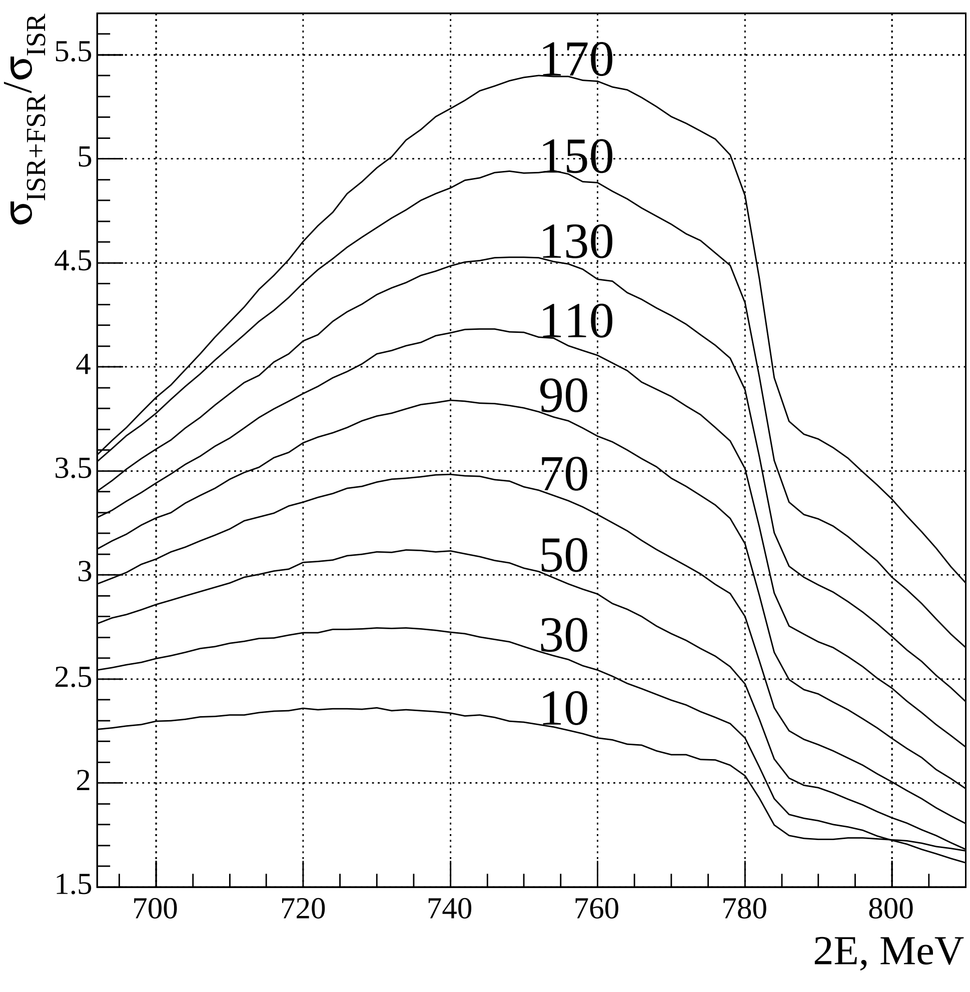

Fig. 9: The ratio of the cross sections with ISR+FSR divided on the cross section with ISR as a function of energy in c.m. frame. The different smooth curves represent this ratio *vs* threshold photon energy (in MeV) to be detected.

just the subject of interest. The comparison of the simulated spectrum with experimental

one can elucidate the discussed problem.

B. Initial state radiation

Let us consider now the initial state radiation (ISR) effects in pion pair production. Performing the calculations similar to the case of muon pair production we have:

$$\frac{d\sigma_{ISR}^{e^+e^-\to\pi^+\pi^-\gamma}}{d\nu} = \frac{\alpha^3}{3s}\frac{1+(1-\nu)^2}{(1-\nu)^2\nu}(l_e-1)(\beta^2-\nu)\sqrt{\frac{\beta^2-\nu}{1-\nu}}|F_\pi(s(1-\nu))|^2, \qquad (35)$$

where $q^2=(q_+ + q_-)^2 = s(1-\nu)$. The calculation results are shown in Fig. 10 with $F_\pi = 1$. Using integrals presented above we can obtain the following expression for the cross section

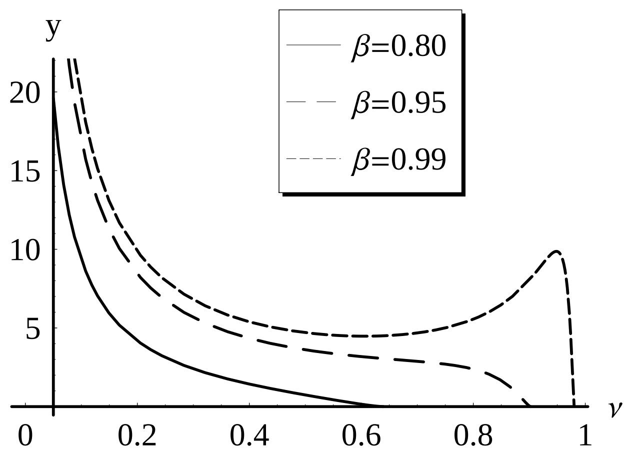

Fig. 10: The distribution pion pairs as a function on ν for ISR. The vertical axis represents the quantity $y = (\alpha^3/3s(l_e-1))^{-1} d\sigma_{ISR}^{e^+e^-\to\pi^+\pi^-\gamma}/d\nu$ (see (35)), horizontal axis - fraction of radiated photon energy.

with hard photon radiation:

$$\sigma_{ISR}^h = \frac{2\alpha^3\beta^3}{3s}(l_e-1)\left\{\ln\frac{1}{\Delta} + 2\ln\left(\frac{2\beta}{1+\beta}\right) - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\right\}, \qquad (36)$$

where $l_e = \ln(s/m^2)$. Here we had assumed the pions to be point-like, i.e $F_\pi = 1$. The sum of the contributions of virtual and soft photon emission has a form:

$$\sigma_{ISR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left\{(l_e-1)\ln\Delta + \frac{3}{4}l_e - 1 + \xi_2\right\}. \qquad (37)$$

The total cross section accounted for initial state radiation can be presented as:

$$\sigma_{ISR}^{e^+e^-\to\pi^+\pi^-\gamma} = \frac{2\alpha^3\beta^3}{3s}\Delta_{ISR}^{\pi^+\pi^-}(\beta), \qquad (38)$$

$$\Delta_{ISR}^{\pi^+\pi^-}(\beta) = (l_e - 1)\left[2\ln\frac{2\beta}{1+\beta} - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\right] + \frac{3}{4}l_e - 1 + \xi_2. \quad (39)$$

Quantity $\Delta_{ISR}^{\pi^+\pi^-}(\beta)$ as a function of β is shown in Fig. 11. In ultra relativistic limit in

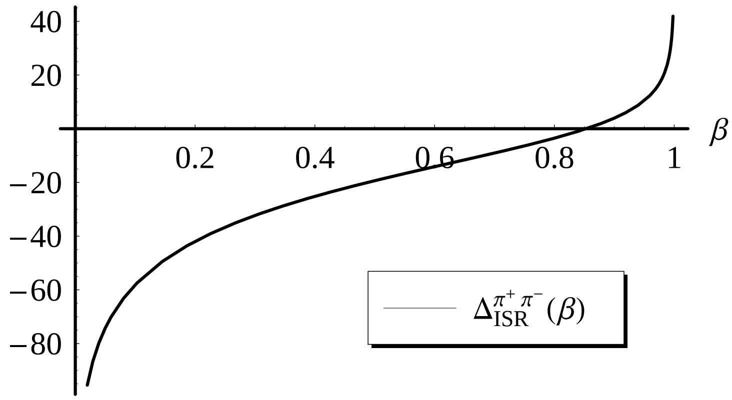

Fig. 11: The dependence of the quantity $\Delta_{ISR}^{\pi^+\pi^-}(\beta)$ (see (39)) on β for ISR.

point-like approximation for pions we have:

$$\sigma_{ISR+FSR}^{e^+e^-\to\pi^+\pi^-\gamma}\Big|_{\beta\to 1} = \frac{2\alpha^3}{3s}\left\{\frac{1}{2}l_e l_\pi - \frac{1}{2}l_\pi + \frac{3}{2}l_e + \frac{1}{6} + \xi_2\right\}, \quad (40)$$

where $l_\pi = \ln(s/M_\pi^2)$.

V. ACCURACY ESTIMATION

The theoretical uncertainties of the cross sections with $\mathcal{O}(\alpha)$ corrections given above are defined by the unaccounted higher order corrections and they are estimated to be at $\sim 0.2\%$ level. Below the main sources of uncertainties which were omitted in the current formulae are listed:

- Weak interactions not considered here arising from replacement of virtual photon Green function by Z-boson one. It results in

$$d\sigma \to d\sigma\left[1 + \mathcal{O}\left(\left(\frac{s}{M_Z^2}\right)^2, \frac{M_\mu^2}{M_Z^2}\right)\right] \quad (41)$$

  which for $\sqrt{s} \le 10 GeV$ is of order or smaller 0.1% in charge-blind experimental setup, when we can omit the $\gamma - Z$ interference contribution.

- Here we systematically omit the terms of order $(m/M_\mu)^2$ compared to 1
$$\mathcal{O}\left(\frac{m^2}{M_\mu^2}\right) \le 0.1\%. \tag{42}$$

- The higher orders contributions (not considered here) can be separated by two classes. First class, leading by large logarithm $l_e = \ln(s/m^2)$, is connected with ISR:
$$d\sigma \to d\sigma\left[1 + \mathcal{O}\left((\alpha/\pi)^2 l_e^2, (\alpha/\pi)^2 l_e\right)\right], \tag{43}$$
$$\mathcal{O}\left((\alpha/\pi)^2 l_e^2\right) \sim 0.2\%, \qquad \mathcal{O}\left((\alpha/\pi)^2 l_e\right) \sim 0.01\%.$$
These kind of contributions can be taken into account by structure function approach as it was done in [18].

- Second class is the higher orders contributions connected with FSR which give
$$d\sigma\left[1 + \mathcal{O}\left(\left(\frac{\alpha}{\pi} l_\beta\right)^2\right)\right], \qquad \mathcal{O}\left(\left(\frac{\alpha}{\pi} l_\beta\right)^2\right) \sim 0.05\%. \tag{44}$$
In ultra relativistic limit $l_\beta \to \ln(s/M_\mu^2)$ they as well can be taken into account by structure function method.

Considering the uncertainty sources mentioned above as independent, we can conclude that the total systematic error of the cross sections with $\mathcal{O}(\alpha)$ RC is less than 0.22 %. However we remind that taking into account of higher order contributions connected with ISR using structure function approach [18] allows one to decrease the total error down to level 0.05 %. $\Delta_{FSR}^{(\mu,\pi)}$ and $\Delta_{ISR}^{(\mu,\pi)}$ are drawn in figures and one can see that corrections to the Born cross sections $(2\alpha/\pi)\Delta$ can reach several percents near threshold.

VI. CONCLUSION

One of possible applications of formulae given above – to be used for normalization purposes at MC simulation. Our results can be used also for improvement of the calculation accuracy of vacuum polarization effects in the virtual photon propagator at low energies not far significantly from threshold production. This calculation, in one's turn, is required to improve the precision of the theoretical prediction for anomalous magnetic moment of muon.

The expressions for the cross sections of $\tau^+\tau^-$ and K^+K^- production are similarly to that for muons and pions. The muon and kaon masses as well as pion form factor should be replaced in the above expressions by the tau and kaon ones, respectively. The cross section being multiplied by the exact Coulomb factor will interpolate the energy dependence of the cross section from the threshold production to the relativistic region.

We do not consider C-odd interference in real and virtual photons emission - it gives zero contribution to the total cross section. As well we do not consider effects of virtual photon polarization operator insertion, it can be found in literature [13, 19].

$\Delta_{FSR}^{(\mu,\pi)}$ and $\Delta_{ISR}^{(\mu,\pi)}$ are drawn in figures and one can see that corrections to Born cross sections $(2\alpha/\pi)\Delta$ can reach several percents near threshold.

In regions where $\beta \sim \alpha$ formulae must be modified [8]. Taking into account, that $\Delta^{(i)}(\beta) \sim \pi^2/4\beta$, $\beta \to 0$, we must replace

$$1 + \frac{2\alpha}{\pi}\Delta^{(i)}(\beta) \to \left(1 + \frac{2\alpha}{\pi}\left(\Delta^{(i)}(\beta) - \frac{\pi^2}{4\beta}\right)\right) f(z)$$

where $f(z) = z/(1 - e^{-z})$ is the Sommerfeld-Sakharov factor, $z = (\pi\alpha/\beta)$. In region there $\beta \ll \alpha$ the formulae must be modified according to [8, 20].

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